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Online International Workshop "Differential Equations and Interdisciplinary Investigations"

Dedicated to the 60th Anniversary of the RUDN University

August 19–21, 2020

PROGRAM & ABSTRACTS

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Wednesday, August 19

*Join the broadcast **here***

9:30–10:00	**Opening**
10:00–10:45	K. Lee, "Dynamical systems from measure theoretical viewpoint. Lecture 1: Why dynamical systems?" (p. 23)
11:00–11:45	W. Jäger, "Can mathematics help to understand and control COVID-19? Lecture 1: Introduction, survey. Basic dynamics of the virus in the host" (p. 14)
12:00–12:45	W. Jäger, "Can mathematics help to understand and control COVID-19? Lecture 2: Development of COVID-19 in the host" (p. 16)
13:00–13:45	H.-O. Walther, "Differential equations with state-dependent delays: some theory, new phenomena, and applications. Lecture 1: The semiflow on the solution manifold" (p. 30)
14:00–15:00	**Lunch**

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15:00–15:45	A. N. Gorban, "Data driven AI: problems and ideas. Lecture 1: Errors of AI and their correctors" (p. 8)
16:00–16:45	C. Bardos, "Vlasov equation: From derivation to quasilinear approximation. Lecture 1" (p. 6)
17:00–17:45	V. Z. Grines, "On interrelation of regular and chaotic dynamics with topology of an ambient manifold. Lecture 1" (p. 11)
18:00–18:45	V. A. Volpert, "Reaction-diffusion equations in biology and medicine. Lecture 1" (p. 25)

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19:00–20:00	**Discussions**

Thursday, August 20

*Join the broadcast **here***

10:00–10:45 **K. Lee**, "Dynamical systems from measure theoretical viewpoint. Lecture 2: Differentiable dynamical systems" (p. 23)

11:00–11:45 **W. Jäger**, "Can mathematics help to understand and control COVID-19? Lecture 3: Growth and spread of the virus infection to a global pandemic — prediction and control" (p. 17)

12:00–12:45 **G. G. Lazareva**, "Mathematical modeling of compressible fluids. Lecture 1: Introduction to compressible flow" (p. 20)

13:00–13:45 **H.-O. Walther**, "Differential equations with state-dependent delays: some theory, new phenomena, and applications. Lecture 2: Complicated motion" (p. 32)

14:00–15:00 **Lunch**

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15:00–15:45 **A. N. Gorban**, "Data driven AI: problems and ideas. Lecture 2: Geometry and topology of data spaces" (p. 8)

16:00–16:45 **C. Bardos**, "Vlasov equation: From derivation to quasilinear approximation. Lecture 2" (p. 6)

17:00–17:45 **V. Z. Grines**, "On interrelation of regular and chaotic dynamics with topology of an ambient manifold. Lecture 2" (p. 12)

18:00–18:45 **V. A. Volpert**, "Reaction-diffusion equations in biology and medicine. Lecture 2" (p. 26)

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19:00–20:00 **Discussions**

Friday, August 21

*Join the broadcast **here***

10:00–10:45	K. Lee, "Dynamical systems from measure theoretical viewpoint. Lecture 3. Recent trends in dynamical systems from measure theoretical point" (p. 23)
11:00–11:45	G. G. Lazareva, "Mathematical modeling of compressible fluids. Lecture 2. Mathematical modeling of compressible fluids" (p. 21)
12:00–12:45	G. G. Lazareva, "Mathematical modeling of compressible fluids. Lecture 3. Applications: The hydrodynamics of interacting galaxies" (p. 21)
13:00–13:45	H.-O. Walther, "Differential equations with state-dependent delays: some theory, new phenomena, and applications. Lecture 3. The impact of state-dependent delay on a periodic orbit" (p. 34)

14:00–15:00 **Lunch**

*Join the broadcast **here***

15:00–15:45	A. N. Gorban, "Data driven AI: problems and ideas. Lecture 3. Logically transparent neural networks" (p. 9)
16:00–16:45	C. Bardos, "Vlasov equation: From derivation to quasilinear approximation. Lecture 3" (p. 6)
17:00–17:45	V. Z. Grines, "On interrelation of regular and chaotic dynamics with topology of an ambient manifold. Lecture 3" (p. 13)
18:00–18:45	V. A. Volpert, "Reaction-diffusion equations in biology and medicine. Lecture 3" (p. 28)

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19:00–20:00 **Discussions**

20:00–20:30 **Closing**

Vlasov Equation: From Derivation to Quasilinear Approximation

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These 3 talks will be devoted to the Vlasov equations with as main thread a mathematical analysis starting from the derivation, based on first principles, to some justification of the quasilinear approximation.

As the solution of a kinetic equation, the Vlasov equations involve quantities $f(x, v, t)$ representing the densities of particles (ions, electrons, or planets) which at the point x and at the time t do have the velocity (or the momentum) v. Since they concern rarified media, the interaction between the particles is described by a mean field potential (average of the action of the other particles on one "tagged" particle).

Therefore compared, for instance, to the Boltzmann equation, the nonlinearity is mild. Moreover, in the case of repulsive interactions considered in these talks, a well defined energy conservation is available. Hence, many results on the existence and stability do exist, cf. [6] for the basic results, [9] for an updated presentation, and [14] for the appearance of singularities in the gravitational problem.

In the derivation from basic principles (classical or quantum dynamics), the Hamiltonian structure is preserved and therefore, if at several steps things are far from trivial, the route to follow is in some sense natural.

Lecture 1

For my first talk I intend as a warm up to derive the macroscopic diffusion from a kinetic equation with a strong relaxation term, then to give an overview of the methods leading from classic or quantum dynamics to the Vlasov equation, describing with more details the use of the Wasserstein metric to derive the Vlasov equations from the BBGKY (Bogoliubov–Born–Green–Kirkwood–Yvon) hierarchy following an extension of the Dobrushin proof [3] proposed in [7].

Lecture 2

The second talk starts with the analysis of the "linearized" Landau damping, following [2], where a functional analysis approach was used in the spirit of Lax and Phillips, [13], or [10], to implement the basics of Landau observations [12]. This contribution emphasizes the need for some regularity (almost analyticity) to consider the genuinely non linear problem as it appears in more modern contributions including [1, 16], and more recently [8].

Lecture 3

With the spectral theory one can approach the issue of the quasi linear approximation as it was done in the classical book of Plasma Physic like Krall and Trivelpiece [11, pp. 514–518]. In some sense it is related to first order correction, as were the original contributions for Fluid Mechanic done by Ellis and Pinsky [4] .

Then to conclude I will (following joint work in progress with N. Besse) derive a large time quasilinear approximation from the original Vlasov equation. It is in this point that appears (in the limit of a fully reversible Liouville equation) the irreversibility.

As it is the case in many companion problems, the limit can be understood in such situation only by considering the stochastic Liouville equation and using the Duhamel series expansion. This what I intend to expose as a conclusion building on a contributions of A. Vasseur and coworkers [5, 15, 17], where it was shown that it is enough to consider the second term in the Duhamel expansion.

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Data Driven AI: Problems and Ideas

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Lecture 1: Errors of AI and Their Correctors

All Artificial Intelligence (AI) systems sometimes make errors and will make errors in the future. These errors must be detected and corrected immediately and locally in the networks of collaborating systems. Real-time re-training is not always viable due to the resources involved and can introduce new mistakes and damage existing skills. One-shot correctors are needed. Correctors are external systems, and the main legacy AI system remains unchanged. The ideal correctors should: be simple, not damage the skills of the legacy system when they work successfully, allow fast non-iterative learning, and allow correction of the new mistakes without destroying previous fixes. If the essential dimensionality of the data is high enough, then the correction problem can be solved by surprisingly simple methods even if the data sets are exponentially large with respect to the dimensionality. This phenomenon is a manifestation of the blessing of dimensionality. The mathematical foundations of these methods are given by stochastic separation theorems that belong to measure concentration theory.

Designing future AI cannot be limited to the development of individual AI systems, but will be naturally extended to the engineering of ecosystems and social networks of AI. Correctors are, at the same time, simple elements of these AI ecosystems and social networks as well as a means of providing cooperation, communication and mutual learning of AI systems, and the division of labour between them.

The lecture presents the theory and applications of AI error correctors and is based on my keynote talk at WCCI2020 (IEEE World Congress on Computational Intelligence, Glasgow, July 20, 2020).

Lecture 2: Geometry and Topology of Data Spaces

Revealing geometry and topology in a finite dataset is an intriguing problem. We present several methods of non-linear data modelling and construction of principal manifolds and principal graphs. These methods are based on the metaphor of elasticity (the elastic principal graph approach). The elastic energy functionals are quadratic and, hence, the computational procedures are not very expensive. The simplest algorithms have the classical expectation/maximization (or splitting) structure.

For the complexity control, several types of complexity are introduced: geometric complexity, structural complexity and construction complexity. The geometric complexity measures how far a principal object deviates from its ideal configuration. The structural complexity counts the number of various elements. It may be represented, for example, by some non-decreasing function of the number of vertices, edges and k-stars of different orders. The construction complexity is defined with respect to a graph grammar as a number of applications of elementary transformations.

Construction of principal graphs with controlled complexity is based on the graph grammar approach and on the idea of pluriharmonic graphs as ideal approximate objects. We present several applications for microarray analysis and visualization of various datasets from genomics, medical and social research. The GIS-inspired methods of datasets cartography are used. In particular, we demonstrate estimation and visualization of uncertainty.

In a series of case studies we compare performance of nonlinear dimensionality reduction to the linear PCA. Nonlinear methods demonstrate better data approximation (as it is expected), better quality of distance mapping (the higher correlation coefficient between the pair-wise distances before and after projection onto the principal object), better quality of point neighborhood preservation and better class compactness.

New open access software library ElPiGraph for construction of principal graphs is illustrated by applications in modern single-cell omics technology.

Lecture 3: Logically Transparent Neural Networks

Explainability of Artificial Intelligence and, specifically, Neural Networks (NN), is a widely recognised problem.

DARPA (Defense Advanced Research Projects Agency, US) launched Explainable Artificial Intelligence (XAI) program and "expected to enable "third-wave AI systems," where machines understand the context and environment in which they operate, and over time build underlying explanatory models that allow them to characterize real world phenomena."

Despite notable successes, the main disadvantages of NN are well known: the risk of overfitting, lack of explainability (inability to extract algorithms from trained NN), and high consumption of computing resources. Too poor NN cannot be successfully trained, but too rich NN gives unexplainable results and may have a high chance of overfitting. Reducing precision of NN parameters simplifies the implementation of these NN, saves computing resources, and makes the NN skills more transparent. Different methods or tools can provide different types of explanation.

We present the basic NN simplification problems and controlled pruning procedures to solve these problems. All the described pruning procedures can be implemented in one framework. The developed procedures, in particular, find the optimal structure of NN for each task, measure the influence of each input signal and NN parameter, and provide a detailed verbal description of the algorithms and skills of NN. The described methods are illustrated by a simple example: the generation of explicit algorithms for predicting the results of the US presidential election.

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On Interrelation of Regular and Chaotic Dynamics with Topology of an Ambient Manifold

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Lecture 1

The structurally stable representatives of discrete dynamical systems with regular dynamics are Morse–Smale diffeomorphisms.

Definition 1. A diffeomorphism $f : M^n \to M^n$ of a smooth closed (compact without boundary) connected orientable n-manifold ($n \geqslant 1$) M^n is called a *Morse–Smale diffeomorphism* if

1. the non-wandering set Ω_f is finite and hyperbolic;
2. for every two distinct periodic points p, q the manifolds W^s_p, W^u_q intersect transversally.

The class of these diffeomorphisms we denote by $MS(M^n)$.

Definition 2. If σ_1, σ_2 are distinct periodic saddle points of a diffeomorphism $f \in MS(M^n)$ for which $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is said to be *heteroclinic*.

- If $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) > 0$, then a connected component of the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is called a *heteroclinic manifold*, and if $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = 1$, then it is called a *heteroclinic curve*.
- If $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) = 0$, then the intersection $W^s_{\sigma_1} \cap W^u_{\sigma_2}$ is countable, each point of this set is called a *heteroclinic point* and the orbit of a heteroclinic point is called the *heteroclinic orbit*.

In this lecture, we state interrelations between the topology of the ambient manifold M^3 and dynamics of a diffeomorphism $f \in MS(M^3)$ whose wandering set does not contain heteroclinic curves (but can contain heteroclinic points). Denote by $MS_*(M^3)$ the class of such Morse–Smale diffeomorphisms. These relations deal with the number

$$g_f = \frac{r_f - l_f + 2}{2},$$

where r_f is the number of the saddle periodic points and l_f is the number of the sink and source periodic points (node points) of the diffeomorphism f.

Theorem 1. *Let f be a diffeomorphism of the class $MS_*(M^3)$ such that Ω_f consists of r_f saddle points and of l_f node points. Then g_f is a nonnegative integer and*

1) if $g_f = 0$, then M^3 is the 3-sphere;

2) if $g_f > 0$, then M^3 is the connected sum of g_f copies of $\mathbb{S}^2 \times \mathbb{S}^1$.

Conversely, for every nonnegative integers r, l, g such that $g = \frac{r-l+2}{2}$ is a nonnegative integer there is a diffeomorphism $f \in MS_*(M^3)$ such that:

a) if $g = 0$, then M^3 is the 3-sphere, and if $g > 0$, then M^3 is the connected sum of g copies of $\mathbb{S}^2 \times \mathbb{S}^1$;

b) the non-wandering set of the diffeomorphism f consists of r saddle points and of l node points.

For main definitions and details of the proof of the above theorem, see [3].

Lecture 2

This lecture is a continuation of the previous one and is devoted to topological classification of closed smooth orientable 3-manifolds admitting Morse–Smale diffeomorphisms whose wandering set do not contain heteroclinic orbits.

Definition 3. A diffeomorphism $f \in MS(M^n)$ is said to be *gradient-like* if the relation $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$ for distinct points $\sigma_1, \sigma_2 \in \Omega_f$ implies $\dim\, W^u_{\sigma_1} < \dim\, W^u_{\sigma_2}$.

The following proposition gives a geometrical interpretation to the property of a homeomorphism to be gradient-like.

Proposition 1. *A diffeomorphism $f \in MS(M^n)$ is gradient-like if and only if it follows from $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \varnothing$ for distinct $\sigma_1, \sigma_2 \in \Omega_f$ that $\dim(W^s_{\sigma_1} \cap W^u_{\sigma_2}) > 0$.*

Thus a Morse–Smale diffeomorphism is gradient-like if and only if it has no heteroclinic points.

Let $MS_0(M^3)$ denote the class of gradient-like diffeomorphisms on the manifold M^3 and $f \in MS_0(M^3)$. The closure $cl\ell$ of any 1-dimensional unstable separatrix ℓ of a saddle point σ of the diffeomorphism f is homeomorphic to the segment consisting of this separatrix and the two points: σ and some sink ω.

Let L_ω be the union of all unstable 1-dimensional separatrices of the saddle points which contain ω in their closures. Since W^s_ω is homeomorphic to $\mathbb{R}^3$ and since the set $L_\omega \cup \omega$ is the union of the simple arcs with the unique common point ω belonging to each arc, we call $L_\omega \cup \omega$ the *frame of 1-dimensional unstable separatrices*.

Definition 4. A frame of separatrices $L_\omega \cup \omega$ is *tame* if there is a homeomorphism $\psi_\omega : W^s_\omega \to \mathbb{R}^3$ such that $\psi_\omega(L_\omega \cup \omega)$ is the standard frame of arcs in $\mathbb{R}^3$. Otherwise the frame of separatrices is called *wild*.

If α is a source of the diffeomorphism f, then a tame (wild) frame $L_\alpha \cup \alpha$ of 1-dimensional stable separatrices is defined similarly.

Let us recall that for $f \in MS_0(M^3)$ one has $g_f = \frac{r_f - l_f + 2}{2}$, where r_f is the number of the saddle periodic points and l_f is the number of the sink and source periodic points (node points) of the diffeomorphism f.

The main aim of this lecture will be a sketch of a proof of the following theorem. Details of the proof can be found in [3].

Theorem 2. *If all the frames of the 1-dimensional separatrices of a diffeomorphism $f \in MS_0(M^3)$ are tame, then the ambient manifold M^3 admits the Heegaard splitting of genus g_f.*

Lecture 3

This lection is devoted to the description of relationships between dynamics of diffeomorphisms of a closed orientable surface M^2 equipped with a metric of constant negative curvature, and their action in the fundamental group $\pi_1(M^2)$.

In the early 1930s, Nielsen introduced a class of homeomorphisms that induce a hyperbolic action in $\pi_1(M^2)$. In 1980, S. H. Aranson and the author of the lecture proposed for any hyperbolic automorphism $\tau : \pi_1(M^2) \to \pi_1(M^2)$ the construction of a homeomorphism $f_\tau : M^2 \to M^2$ whose non-wandering set has an invariant subset Ω_0 being the intersection of two transversal geodesic laminations uniquely defined by τ. The restriction of f_τ to the set Ω_0 has a countable set of saddle periodic points of non zero topological entropy, which is minimal among entropies of all homeomorphisms homotopic to f_τ. Moreover, f_τ is homotopic to the pseudoanosov homeomorphism introduced by W. Thurston in 1970s.

As an application of the results described above, the lecture will include topological classification of structurally stable diffeomorphisms on M^2 whose non-wandering set consists of a perfect widely disposed one-dimensional attractor and a finite number of source periodic orbits [1], [2]. Moreover, the results will be discussed on topological classification of structurally stable diffeomorphisms belonging to different homotopic Nilsen-Thurston classes. For information and history of questions connected with the topic, see book [3].

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Can Mathematics Help to Understand and Control COVID-19?

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Lecture 1

0. Introduction, Survey

1. **Main Aims of the Lectures**
   1.1 Mathematical and Computational Sciences are necessary to master the challenges
   1.2 Requirements from epidemiologists and decision makers
2. **History of the Virus SARS-CoV-2 and COVID-19**
   2.1 Dynamics and spread of the virus and the infection
   2.2 Damages of the pandemic and challenges
3. **Typical Features of the Dynamics of SARS COV2 and COVID-19**
   3.1 Dynamics in the host
   3.2 Dynamics in the population
4. **The Hidden and Unpredictable Virus**
   4.1 Hidden, unpredictable virus
   4.2 Virus profile
   4.3 Uncertainties in its effects on the organism, genetic instabilities
5. **COVID 19 — a Virus Sepsis**
   5.1 Sepsis — systemic disease leading to multiple organ failure
   5.2 Hypoxia, inflammation, disordered immune response
6. **Challenges:**
   6.1 Medical challenges
   6.2 Biomedical and biotechnological challenges
   6.3 Economical and financial challenges
   6.4 Political, social, cultural challenges
   6.5 Scientific challenges
7. **Challenges to Mathematics and Information Technology**

7.1 Data collection and analysis

7.2 Prognostic and strategy of control of the evolution of the disease in individual infected and in the populations

7.3 Quantitative approaches for design, production and applications of tests, vaccine and therapies and medications

8. **Mathematical and Computational Concepts and Tools in Demand**

8.1 Data analysis: statistical and machine learning methods in time, space and state space

8.2 Discrete and continuous, stochastic and deterministic modelling of physical, chemical, biological, medical, technological and social processes, complex multiscale media, with nonlinear interactions

8.3 System reduction and methods bridging scales method — to reduce complexity

8.4 Effective, reduced systems — mainly coupled systems of differential-functional equations in very often evolution dependent domains, examples

I. Basic Dynamics of the Virus in the Host

1. **Selected Biological and Medical Facts**

1.1 Virus SARS COV 2

1.2 COVID-19 a Viral Pneumonia

1.3 Hypoxia, inflammation, cytokine storm — start of a virus sepsis

2. **Molecular and Microbiology of the Virus**

2.1 Structure of the virus

2.2 Spike Glycoprotein- hot topic of mathematical research

2.3 The cycle of the virus

3. **Entry of the Virus to the Cell**

3.1 Docking of the virus to cell, the role of the Spike Glycoprotein

3.2 A mathematical model for penetration (project for HIV-virus)

4. **The Process of Translation and Replication**

4.1 Mathematical modelling of the processes

4.2 Identification of “checkpoints” and optimal treatment strategies

5. **Effects of the Virus Infections on a Single Cell**

5.1 Changes of the structure

5.2 Changes of the cellular processes

6. **Interactions with the Immune System**

6.1 Formation of antibodies — the role of B and T cells

7. **Can Mathematics Contribute to the Human Responses to Sars-Cov-2 and Covid-19?**

    7.1 Design and implementation of test

    7.2 Comments to the situation in producing and implementing vaccines and medications

Lecture 2

II. Development of COVID-19 in the Host

1. **Start of the Disease**

    1.1 Infection of the lung and impacts

    1.2 Consequences for the organism, hypoxia, spreading of material, start of a virus sepsis

    1.3 The role of ACE2

2. **Hypoxia, Hypoxemia and Impacts**

    2.1 Central role of oxygen and energy supply

    2.2 Hypoxia in tissue — in blood

3. **Cytokine Storm and Impacts**

    3.1 Ordered and disordered cell signaling

    3.2 Inflammatory and anti-inflammatory effects

4. **Processes at Endothelial and Epithelial Layers**

    4.1 Crucial cell layers gating and controlling the transmission between compartments

    4.2 Modelling of the processes on micro- and meso-level

    4.3 Multiscale approaches and derivations of approximating system and corrector terms.

5. **Blood Clotting**

    5.1 Clotting and Infarcts

    5.2 Processes and important factors, impact of the infection: Tissue Factor, S Protein, Van Willibrand-Factor

    5.3 Modelling and simulation of thrombus formation in arteries

    5.4 Open problems: veins, capillary systems

6. **Inflammation**

    6.1 General situation — virus specific situation

6.2 Modelling and simulation of basic stages

7. **Oxygen and Energy Supply and Dysfunctions**

   7.1 The central role of mitochondria

   7.2 Modelling of oxidative respiration and dysfunction, production of ATP and ROS

8. **The Severe Stage of COVID-19 as a Virus Sepsis**

   8.1 Virus infections of the organs

   8.2 Specific features and their challenges to modelling, simulation and control of a virus sepsis dynamics

9. **Tests, Vaccination, Therapies**

   9.1 Mathematical and computational methods for antigen and antibody tests

   9.2 Mathematics for development of an optical test system (Super-Resolution Light Microscopy)

   9.3 Modelling and simulations relevant processes for different vaccines and medications under conditions similar to those in the human body (digital twin concept)

10. **Summary: Math Challenges and Potential**

   10.1 Modelling processes on micro-meso-macroscale, deterministic and stochastic approaches

   10.2 Linking the scales and the approaches, reduction of complex model systems to efficient systems, that can be calibrated and computed and provide answers to posed questions

Lecture 3

III. Growth and Spread of the Virus Infection to a Global Pandemic — Prediction and Control

1. **Sketch of the General Structure of an Epidemic / Pandemic**

   1.1 Basic elements setting up model for SARS-CoV-2

   1.2 Coupling of virus dynamics in a single host and in a population

   1.3 Gaps in the current approach of modelling the virus epidemic

2. **Profile of the Virus**

   2.1 List feature of the SARS-CoV-2 import for growth and spread in populations

   2.2 Relevant features for model development

3. **Profiles of the Population — GHS Global Health Security Index**

3.1 Indices characterizing the epidemical risks

3.2 Validation of the GHS index based on Pandemic data

4. **Can the Complexity in Describing Infection-Processes and the Uncertainties in Predicting the Evolution of a Pandemic be Reduced?**

   4.1 The wave of the corona virus outbreak in USA

   4.2 Message: simple generalizations of SIR- model cannot not reflect the dynamics of COVID

   4.3 Decision makers are dependent on science, especially mathematics, in this uncertainty

   4.4 The role of Robert Koch Institute (RKI), Germany, nationally, internationally

5. **Scenarios of Reactions to Master the Pandemic**

   5.1 Mitigation measures

   5.2 The concept of herd immunity

   5.3 Critical remarks

6. **An Interim Assessment of the Demands on Mathematics and the Computational Sciences**

   6.1 Deficits of current models

   6.2 Suggestion for improvement

7. **Mathematical Model, Used by RKI to Estimate the Reproduction Factor Ro (Example 1)**

   7.1 Formulation of the model (generalization of SIR-Model)

   7.2 Presentation of results, critical comments

8. **Mathematical Model for COVID-19 in China (B. Ivorra et al., 30.04.2020) (Example 2)**

   8.1 Formulation of the model (generalization of SIR-Model)

   8.2 Discussion of the results

9. **Mathematical Model to Study HERD Immunity (T. Britton et al., 23.06.2020) (Example 3)**

   9.1 Formulation of the model (generalization of SIR-Model)

   9.2 Assumptions

   9.3 Critical comments concerning herd immunity

10. **Modelling, Simulations and Optimization of Interventions**

    10.1 Testing

    10.2 Vaccination

10.3 Introducing and lifting restrictive measures

11. **Summary and Outlook: Math Challenges and Potential**

11.1 Summary and outlook for Chap. III and the main topic of the lecture series.

11.2 Can Mathematic help to recover from the lateral damage of the Pandemic?

Mathematical Modeling of Compressible Fluids

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Lecture 1: Introduction to Compressible Flow

Applied mathematics covers a very wide area of knowledge. In this mini-course, we will consider mathematical modeling of problems in continuum mechanics.

Computer simulations are carried out to understand the consequences of fundamental physical laws, help interpret experiments, provide detailed information which is difficult to measure, design and predict new experiments. Well-known triad of Academician A.A.Samarskii is: *Model — Algorithm — Program*. Academician A.N.Konovalov (Novosibirsk) expanded the concept: *Experiment — Mathematical Model — Discrete Model — Algorithm — Program — Supercomputer*.

To understand all the stages of mathematical modeling, you need to get basic knowledge of continuum mechanics. This lecture covers the basic concepts of Fluid Mechanics. How compressible are fluids? Is it same for liquids and gases? What is the difference between an isothermal compression and isentropic compression? What is the speed of sound? How to calculate the speed of sound?

In this lecture, we will learn about the characteristics of fluids used in mechanical systems (viscosity, viscosity index, compressibility and hydraulic fluid). The description of gas movement in the framework of the phenomenological approach is connected with the ideas about the average values that characterize its state. When deriving the main equations of the mathematical model, it is postulated that the average volume density, velocity, and other values tend to a certain limit when the volume is pulled to a point. This assumption is justified by the fact that the sizes of volumes containing a sufficiently large number of molecules can be chosen small in comparison with the typical scales of the studied phenomena. But the small-scale movement of a large number of molecules within a small volume must also be taken into account in the simulation. The theory uses an approximate description of small-scale processes. This introduces some average parameters that characterize the state of the gas. A number of relationships between these parameters are the result of General physical laws.

Other relations (equations of state of specific media) arise as a result of generalization of experimental data. It is known that a gas located in a fixed volume, with fixed external parameters and in the absence of energy exchange with external bodies, after some time (relaxation time) comes to an equilibrium state. The smaller the volume, the shorter the relaxation time. Therefore, when modeling processes with the characteristic time of change of average values much longer than the relaxation time, it can be assumed that a small volume of gas is in the state of equilibrium with fixed external parameters at each time. A continuous change in its main characteristics is interpreted as a quasi-static transition from one equilibrium state to another.

Lecture 2: Mathematical Modeling of Compressible Fluids

The equations of hydrodynamics are nonlinear. This means that only in very rare cases can an analytical solution be found. Thus, the use of numerical methods in solving problems of hydrodynamics is absolutely necessary.

The speed and memory of computers increase fantastically fast creating more and more "smart" algorithms — all this led to the rapid development of computational fluid dynamics. In the near future, computer calculations will replace expensive experiments in wind tunnels. Today, software for calculating the flow of liquids and gas is not only a science but also a commercial product that is actively sold, purchased, and used in many industries.

Macroscopic movements of the compressed gas are described by the system of Navier–Stokes equations. These equations can be derived from the laws of conservation of mass, momentum, and energy. If we neglect the viscous (diffusive) terms included in these equations, leaving only the terms responsible for convective transport and normal pressure forces, then we come to the system of Euler's equations.

The main difficulties of numerical simulation arise precisely when solving these inviscid equations. A method suitable for solving Euler equations can usually be extended quite simply to the Navier–Stokes equations. For example, you can approximate the diffusion terms with central differences. When solving the Navier–Stokes equations (especially for large Reynolds numbers), however, there are difficulties. These difficulties are associated with the presence of thin boundary and free shear layers and turbulence of the flow.

In this lecture, the governing equations for a flow field are derived. The fluid flow field can be described in various ways. Both Lagrangian (moving with the flow) and Eulerian (fixed in the flow) specifications are explored. The substantial (or total) derivative appears frequently when deriving the governing equations. We can imagine that we hop on a fluid element as it is moving through a flow field, and we look at how the properties of the fluid element change. This is fundamentally different from sitting still and watching the flow move through a fixed point, and then seeing how the properties change at that point.

The difference between the conservative case and non-conservative one is investigated. The distinction between integral form and differential form are described. Discrete models based on the solution of hyperbolic equations are considered. The concepts of dispersion and dissipation of schemes are introduced. We consider only uniform grids for simplicity. These grids are the easiest way to get started with coding your own compressible fluids dynamics routines.

Lecture 3: Applications: The Hydrodynamics of Interacting Galaxies

There are very interesting applications of gas-dynamic equations in astrophysics. In this lecture, you will learn the model of the Central collision of two galaxies, numerical methods for hydrodynamics, verification of a numerical method, and parallel implementation.

This project had three motivations: physics, supercomputers, and design concept. As Prof. Alexander Tutukov (INASAN, Moscow) said, "*The movement of galaxies in dense clusters turns the collisions of galaxies into an important evolutionary factor*." The movement of galaxies in dense clusters turns the collisions of galaxies into an important evolutionary factor because during the Hubble time an ordinary galaxy may suffer up to 10 collisions with the galaxies of its cluster.

Observational and theoretical study of interacting galaxies is an indispensable method for understanding their properties and evolution. A collision may result in the destruction of the galaxies, their coalescence, the conservation of the stellar components, and the destruction of gas components or the conservation of the stellar components with the formation of a new galaxy from their gas components.

The gas component plays a major role in the scenario of the collision of galaxies. Thus, it is necessary to simulate the collision of galaxies by means of the hydrodynamical approach. The model is based on the solution of the equations of gas dynamics, supplemented by the equation for gas inner energy, the Poisson equation for the gravitational potential, and the cooling function. The stellar component of the galaxies is simulated by the central body that brings its contribution to the common value of the potential.

In this lecture, my hydrodynamical code for the numerical simulation of the collision of the gas components of galaxies is described. The code is based on the Fluids-in-Cells method with the Godunov-type scheme at the Eulerian stage. Also, the velocity correction is employed at the Lagrangian stage and energy imbalance is minimized.

The performance of the code is shown by the simulation of the collision of gas components of two similar disk galaxies in the course of the central collision of the galaxies in the polar direction. As a result of the numerical simulations within the model, we succeeded in determining the main scenarios of the collision of galaxies. At low velocities, both galaxies and their gas components coalesce. At high velocities, the massive stellar components of galaxies dissipate almost freely, leaving their gaseous components slowed down and heated by the collision. If the common gas component of the colliding galaxies cools down to the virial temperature, a new galaxy is formed from the two. With the high collision velocity, the gas component has no time to cool and therefore the gas dissipates in the intergalactic medium.

Dynamical Systems from Measure Theoretical Viewpoint

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Lecture 1: Why Dynamical Systems?

The subject of dynamical systems concerns the evolution of systems in time. In continuous time, the systems may be modeled by ordinary differential equations or partial differential equations; in discrete time, they may be modeled by difference equations or iterated maps. The emphasis of dynamical systems is the understanding of geometrical properties of orbits and long term behavior. In the first lecture I will present on the motivation, classification, and basic knowledge of dynamical systems.

Lecture 2: Differentiable Dynamical Systems

A dynamical system is briefly a “flow,” or “iteration.” Being a rich mathematical theory, dynamical systems has a strong background of science and technology. Several branches have appeared successively, such as topological dynamical systems, ergodic theory, differentiable dynamical systems, measurable dynamical systems, random dynamical systems, Hamiltonian systems, etc. Nevertheless the division is not rigorous and the branches overlap. By differentiable dynamical systems is usually meant the one concerning structural stability, hyperbolicity, genericity, etc, developed since the 1960’s. In the second lecture I attempt to present some dynamic properties of the hyperbolic sets of flows induced by C^1 vector fields on compact smooth manifolds, such as expansivity, shadowing property, topological stability, etc.

Lecture 3: Recent Trends in Dynamical Systems from Measure Theoretical Viewpoint

In the last lecture, I will discuss some recent and ongoing works on the dynamics of flows with various expansive measures. In particular, we present a measurable version of Smale’s spectral decomposition theorem for flows. More precisely, we prove that if a flow ϕ on a compact metric space X is invariantly measure expansive on its chain recurrent set $CR(\phi)$ and has the shadowing property on $CR(\phi)$, then ϕ has the spectral decomposition, i.e., the nonwandering set $\Omega(\phi)$ is decomposed by a disjoint union of finitely many invariant and closed subsets on which ϕ is topologically transitive. Moreover, we show that if ϕ is invariantly measure expansive on $CR(\phi)$, then it is invariantly measure expansive on X. Using this, we characterize the measure expansive flows on a compact smooth manifold via the notion of Ω-stability.

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Reaction-Diffusion Equations in Biology and Medicine

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Lecture 1

1.1. Introduction to the theory of reaction-diffusion waves

We will begin these lectures with a short introduction to the theory of reaction-diffusion equations. We will recall the main notions and how this theory was developed. We consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) \tag{1}$$

on the whole axis with a sufficiently smooth function $F(u)$ such that $F(0) = F(1) = 0$. Travelling wave solution of this equation is a solution of the form $u(x,t) = w(x - ct)$, where c is the wave speed. It is an unknown constant which should be found together with the function $w(\xi)$ satisfying the equation

$$w'' + cw' + F(w) = 0 \tag{2}$$

and such that

$$w(-\infty) = 1, \quad w(\infty) = 0. \tag{3}$$

Systematic theory of reaction-diffusion waves begins in the 1930s with the works by Fisher [3] and KPP [4] in population dynamics, by Zeldovich and Frank-Kamenetskii in combustion theory [12], and by Semenov in chemical kinetics [7], but the first works by Mikhelson [6] and Luther [5] appeared several decades earlier.

Existence and stability of reaction-diffusion waves depend on the nonlinearity $F(u)$. It is convenient to classify the functions $F(u)$ according to the stability of the stationary points $u = 0$ and $u = 1$ of the equation $du/dt = F(u)$. In the bistable case, both of them are stable; in the monostable case, one point is stable while another one is unstable; in the unstable case, both of them are unstable.

1.2. Existence of waves

Existence of solutions of problem (2), (3) can be studied by the phase plane analysis for the corresponding system of first-order equations. In the monostable case, under the assumption that $F(u) > 0$ for $0 < u < 1$, monotonically decreasing solutions of this problem exist for all values of speed c greater than or equal to some minimal speed $c_0 > 0$. We will see below that the monotone waves are stable, while non-monotone waves are unstable. The latter exist for all positive c. If in addition $F'(u) \leqslant F'(0)$, then $c_0 = 2\sqrt{F'(0)}$.

Without the assumption that the function $F(u)$ is positive in the interval $[0,1]$, a solution of problem (2), (3) may not exist. If it exists, then it is possible to affirm that the $[0,1]$-waves, that is, solutions of equation (2) with limits (3), exist in some interval $[c_0, c_1)$ of speeds. If such waves do not exist, then the solution of the Cauchy problem for equation (1) is described by systems of waves (see below).

In the simple bistable case where $F(u) < 0$ for $0 < u < u_0$, $F(u) > 0$ for $u_0 < u < 1$, and some $u_0 \in (0,1)$, there exists a solution of problem (2), (3) for a unique value of speed c. In the general bistable case, if the $[0,1]$-wave exists, then the speed is unique. If it does not exist, then, as before, there are systems of waves. Finally, in the unstable case, such wave does not exist.

1.3. Existence of pulses

Pulse is a positive stationary solution of equation (1) with the zero limits at infinity. For the scalar equation, existence of such solutions can be studied by the phase plane analysis of the first-order system of equations

$$w' = p, \quad p' = -F(w), \tag{4}$$

or even found analytically. Since $(0,0)$ is a stationary point of system (4), pulse corresponds to a homoclinic orbit starting and ending at this point. It exists in the bistable case if $\int_0^1 F(u)du > 0$.

Lecture 2

2.1. Spectrum and stability

There exist different types of wave stability. A solution $u(x,t)$ of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u) \tag{5}$$

(with a sufficiently smooth function $F(u)$ such that $F(0) = F(1) = 0$) satisfying some initial condition $u(x,0) = u_0(x)$ converges to a wave $w(x)$ in form and in speed if there exists a function $m(t)$ such that $u(x,t) \to w(x - m(t))$ uniformly in $\mathbb{R}$, and $m'(t) \to c$. Convergence in form and in speed is equivalent to the convergence on the phase plane. This convergence (in other terms) and the corresponding method of studying it were introduced in KPP [4]. The uniform convergence implies that $u(x,t) \to (w - ct - h)$ uniformly in $\mathbb{R}$ for some constant h. Convergence in form and in speed follows from the uniform convergence but the opposite may not be true.

In the monostable case, a solution of the Cauchy problem converges to one of the $[0,1]$-waves depending on the initial condition $u_0(x)$. Namely, if $u_0'(x)/u_0(x) \to -\lambda$, where $\lambda = c/2 - \sqrt{c^2/4 - F'(0)}$, and $\underline{\lim}_{x\to-\infty} u_0(x) > 0$, then the solution converges in form and in speed to the wave with the speed $c \geqslant c_0$. If $\lambda \leqslant c_0/2 - \sqrt{c_0^2/4 - F'(0)}$, then the convergence occurs to the wave with the minimal speed c_0. More general results are also known (see [9] and the references therein). In applications, initial conditions such that $u_0(x) \equiv 0$ for x sufficiently large are often considered. In this case, the solution converges to the wave with the minimal speed. The uniform convergence in the monostable case occurs under some additional conditions.

In the bistable case, the $[0,1]$-wave is globally asymptotically stable in the sense of uniform convergence if $F'(0) < 0, F'(1) < 0$ and for a large class of initial conditions.

The convergence in form and in speed occurs without the last condition on the derivatives.

In general, monotone waves for the scalar reaction-diffusion equations are stable, and non-monotone waves are unstable. This can be seen from the analysis of the spectrum. In the bistable case, it can be easily verified that the linearized operator $Lv = v'' + cv' + F'(w(x))v$ has the zero eigenvalue with the corresponding eigenfunction $v_0(x) = -w'(x)$. If $w(x)$ is a monotonically decreasing function, then the eigenfunction $v_0(x)$ is positive. Therefore, 0 is the eigenvalue with the maximal real part (the principal eigenvalue) [8], and all other points of the spectrum lie in the left-half plane. Such structure of the spectrum provides asymptotic stability of waves with shift with respect to small perturbations. If the wave is non-monotone, then the eigenfunction $v_0(x)$ is alternating. Hence, 0 is not the principal eigenvalue, and there is a positive eigenvalue of the operator L_0. Thus, the wave is unstable. In the monostable case, the situation is more complex because of the essential spectrum but the result about stability of monotone waves (in certain sense) and instability of non-monotone waves remains valid. Similar to non-monotone waves, pulses are unstable.

2.2. Systems of waves

If the $[0,1]$-waves do not exist, then behavior of solutions of the Cauchy problem is determined by systems of waves. In order to explain this notion, consider the following example. Suppose that $F(0) = F(u_0) = F(1) = 0$ for some $u_0 \in (0,1)$, and $F'(0), F'(u_0), F'(1) < 0$. Hence, we can consider the bistable $[0,u_0]$-wave and another bistable $[u_0,1]$-wave assuming that they exist. Denote by c_1 the speed of the former and by c_2 of the latter. If $c_1 > c_2$, then there are two waves propagating one after another with different speeds, and the solution $u(x,t)$ converges to a two-step function formed by these waves. If $c_1 < c_2$, then the two waves merge, and there is a single $[0,1]$-wave. Such solutions were first studied in combustion theory. In the mathematical context they were introduced and studied in [1,2] (called minimal decomposition of waves there). For general functions $F(u)$ they were studied in [10,11].

2.3. Systems of equations

Consider now equation (5) with the vector variables $u = (u_1, ..., u_n)$, $F = (F_1, ..., F_n)$. This system of equations is called a monotone system if the following inequalities

$$\frac{\partial F_i}{\partial u_j} > 0\ , \quad i,j = 1,...,n, \quad i \neq j \tag{6}$$

are satisfied for all $u \in \mathbb{R}^n$. This is a class of systems for which the maximum principle and comparison theorems, conventionally used for the scalar equation, remain valid. These properties of monotone systems provide the results on the wave existence and stability similar to the results presented above for the scalar equation [8,9]. Furthermore, the minimax representation of the wave speed and the results on the systems of waves are also applicable for the monotone systems.

For a more general class of locally monotone systems, inequalities (6) are supposed to hold only on the surfaces $F_i = 0$. In this case, the maximum principle is not applicable but it is still possible to prove wave existence in the bistable case.

Lecture 3

3.1. Applications in biology and medicine

There are numerous applications of reaction-diffusion equations in biology and medicine. We can cite some examples: invasion of populations and prey-predator models in population dynamics, emergence and evolution of biological species in ecological models, epidemic spreading in epidemiological models. In biomedical models: tumor growth, infection spreading in the tissue, waves of electric potential in the brain cortex, and so on. We will consider two examples in more detail.

3.2. Population dynamics with nonlocal consumption of resources

The logistic equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + au(1 - u/K) \tag{7}$$

for the population density $u(x,t)$ describes random motion of the individuals in the population and their reproduction according to the logistic law. The reproduction rate is proportional to the population density u and to available resources $(1 - u/K)$.

In the case of nonlocal consumption of resources, the conventional logistic term is replaced by the expression $u(1 - J(u))$, where $J(u) = \int_{-\infty}^{\infty} \phi(x-y)u(u,t)dy$. The kernel $\phi(x-y)$ determines the efficiency of consumption of resources depending on the distance $|x-y|$. This model is developed to describe the emergence of biological species in the process of their evolution. From the mathematical point of view, they correspond to the propagation of periodic waves. In the case of global consumption of resources, available resource are proportional to $(1 - I(u))$, where $I(u) = \int_{-\infty}^{\infty} u(u,t)dy$. Contrary to the conventional local equation, in the bistable case here, the pulse solution becomes stable. Such solutions describe population density for stable persistent biological species.

3.3. Models of viral infection

Spatial model of viral infection

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + au(1 - u/K) - f(u_\tau) \tag{8}$$

for the virus density $u(x,t)$ describes infection spreading in the tissues of the organism. Here the last term in the right-hand side characterizes virus elimination by the immune response, $u_\tau = u(x, t-\tau)$. Time delay τ corresponds to the duration of clonal expansion of immune cells in the adaptive immune response. We will discuss dynamics of solutions of delay reaction-diffusion equations and their biomedical interpretations.

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Differential Equations with State-Dependent Delays: Some Theory, New Phenomena, and Applications

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Lecture 1: The Semiflow on the Solution Manifold

Delay differential equations with state-dependent delay arise in applications from physics to the life sciences, information technology, and mechanical engineering [4], and they are of interest as mathematical objects since they are not covered by the theory of retarded functional differential equations as it has been presented in monographs through the decades since 1940, up to [2,3]. Before the millenium it was even unknown how to linearize equations with state-dependent delay at an equilibrium solution.

A toy example of a differential equation with state-dependent delay is

$$x'(t) = g(x(t-d)), \quad d = \delta(x(t)), \tag{1}$$

with continuously differentiable functions $g : \mathbb{R} \to \mathbb{R}$ and $\delta : \mathbb{R} \to [0, r]$ for some $r > 0$.

The lecture begins with a look at the by now familiar initial value problem

$$x'(t) = f(x_t) \quad \text{for} \quad t > 0, \tag{2}$$

$$x_0 = \phi \in U, \tag{3}$$

for retarded functional differential equations where ϕ belongs to the Banach space $C_n = C([-r, 0], \mathbb{R}^n)$ of continuous maps $[-r, 0] \to \mathbb{R}^n$, the segment x_t of the solution is defined by $x_t(s) = x(t+s)$ for $-r \leqslant s \leqslant 0$, and the map $f : C_n \supset U \to \mathbb{R}^n$ is at least locally Lipschitz continuous. Initial value problem (2) and (3) is well-posed for solutions $x : [-r, t_e) \to \mathbb{R}^n$, $0 < t_e \leqslant \infty$, which are continuous and differentiable for $t > 0$. This result applies to differential equations with constant delay, discrete or distributed. But it fails for equations with state dependent delay like Eq. (1). The reason is a specific lack of smoothness which is closely related to the fact that the evaluation map

$$ev : C_1 \times [-r, 0] \ni (\phi, s) \mapsto \phi(s) \in \mathbb{R}$$

is only continuous but not locally Lipschitz continuous. Notice that Eq. (1) has form (2) for f given by

$$f(\phi) = g(\phi(-\delta(\phi))) = [g \circ ev \circ (id \times (-\delta))](\phi).$$

There are examples of equations with state-dependent delay for which solutions are not uniquely determined by initial data which are only continuous. So one is led to look for another state space, different from C_n, on which the initial value problem is well-posed. Moreover, linearization should be possible, which means that solutions should be differentiable with respect to their initial data. The search for a suitable state space begins with the observation that the restriction ev_1 of the evaluation map to the product $C^1_1 \times (-r, 0)$, with the Banach space $C^1_1 = C^1([-r, 0], \mathbb{R})$ of continuously differentiable functions $[-r, 0] \to \mathbb{R}$, is continuously differentiable. We have

$$D\, ev_1(\phi, s)(\hat{\phi}, \hat{s}) = \hat{\phi}(s) + \hat{s}\, \phi'(s),$$

and maps $f : U \to \mathbb{R}^n$ with domain U open in C^1_n which represent differential equations with state-dependent delay like the toy example Eq. (1) become continuously differentiable. It will then be explained how the idea of solutions of Eq. (2) with segments x_t, $0 < t < t_e$, in $U \subset C^1_n$ leads to the necessary condition

$$\phi'(0) = f(\phi)$$

for the initial values $\phi = x_0 \in U$. Only for initial values in the set

$$X_f = \{\phi \in U : \phi'(0) = f(\phi)\}$$

one can expect solutions with all segments in C^1_n. The set X_f will not be open (unless $X_f = \emptyset$) — this is very much in contrast to other initial value problems, beginning with ordinary differential equations. The question about the nature of the set X_f arises. At this point another observation comes into play: Often continuously differentiable maps f which represent differential equations with state-dependent delay have the additional smoothness property that

(e) *each derivative $Df(\phi) : C^1_n \to \mathbb{R}^n$, $\phi \in U$, has a linear extension $D_e f(\phi) : C_n \to \mathbb{R}^n$, and the map*

$$U \times C_n \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

Property (e) is a version of the notion of being *almost Fréchet differentiable* which was introduced by Mallet-Paret, Nussbaum, and Paraskevopoulos [10]. With regard to the verification of property (e) in examples we point out that the formula for the derivative of ev_1 makes sense also for $\hat{\phi} \in C_n$.

Having property (e) it will be shown that the set X_f if non-empty is a continuously differentiable submanifold of codimension n in the space C^1_n. Its tangent spaces are given by

$$T_\phi X_f = \{\chi \in C^1_n : \chi'(0) = Df(\phi)\chi\}.$$

For initial data $\phi \in X_f$ the initial value problem is well-posed. The solutions $x = x^\phi$, or better, the curves $t \to x^\phi_t$, $\phi \in X_f$, generate a continuous semiflow $S : (t, \phi) \mapsto x^\phi_t$ on X_f, with all solution operators $S(t, \cdot) : \phi \mapsto x^\phi_t$, $t \geqslant 0$, continuously differentiable [4, 20]. The proof of this result involves a contraction argument for which property (e) is crucial.

Now linearization is possible. As in, say, ordinary differential equations the derivatives (linearizations)

$$D_2 S(t, \phi) : T_\phi X_f \to T_{S(t,\phi)} X_f, \quad \text{for} \quad \phi \in X_f \quad \text{and} \quad 0 < t < t_\phi,$$

of the solution operators are given by the solutions $v = v^{\phi,\chi}$ of linear variational equations

$$v'(t) = Df(S(t,\phi))v_t$$

with initial data $v_0 = \chi \in T_\phi X_f$,

$$D_2 S(t,\phi)\chi = v_t^{\phi,\chi}.$$

Having smoothness of solution operators the local tools of dynamical systems theory become available, namely, the principle of linearized stability, local invariant manifolds at equilibria, and Poincaré return maps on transversals of periodic orbits.

Linearization at stationary points leads to the same linear variational equations as the classical theory would do. For toy example (1) with, say, $g(0) = 0$ the variational equation along the trivial solution $t \mapsto 0$ looks the same as for the equation with delay frozen at equilibrium,

$$x'(t) = g(x(t - \delta(0))). \tag{4}$$

On the level of linearization the stability properties of the zero solution $t \mapsto 0$ of Eq. (1) are the same as for $t \mapsto 0$ considered as solution of Eq. (4) with constant delay.

For periodic orbits, however, state-dependent delay does affect the stability properties on the level of linearization. This will be made precise in lecture 3 about a recent case study [11, 12].

Last not least one may wonder what kind of manifold the *solution manifolds* X_f are, topologically complicated or not? Recently we found that for large classes of differential equations with state-dependent delay, among them the toy example for $\delta(\xi) > 0$ everywhere, they are simply diffeomorphic to open sets in a subspace $H \subset C_n^1$ of codimension n [22].

Lecture 2: Complicated Motion

What is the impact of state-dependent delay on a dynamical system? In particular, how can solution behaviour change if in a differential equation with constant delay this delay is replaced with a variable, state-dependent delay? The lecture presents recent results which give first answers to these questions: State-dependent delay alone may cause complicated solution behaviour, that is, chaotic motion as introduced by Shilnikov for ordinary differential equations, and complicated motion of a new kind.

We abbreviate $C = C_1 = C([-r, 0], \mathbb{R})$ and $C^1 = C_1^1 = C^1([-r, 0], \mathbb{R})$, for $r > 0$.

First we review results on chaotic motion for equations with a constant time lag, like

$$x'(t) = f(x(t-1)) \tag{5}$$

with $f : \mathbb{R} \to \mathbb{R}$. Here it is only the shape of the function f which may cause complicated motion. For certain f which represent negative feedback with respect to the zero solution by means of the condition

$$\xi\, f(\xi) < 0 \quad \text{for all} \quad \xi \neq 0$$

and which are not monotonic it was shown in [1, 5–7, 18] that chaotic motion as observed by Poincaré [14] for ordinary differential equations is present, close to a solution $h : \mathbb{R} \to \mathbb{R}$ which is homoclinic to an unstable periodic orbit $\mathcal{O} \subset C$ (C with $r = 1$ here),

$$h_0 \notin \mathcal{O} \quad \text{and} \quad h_t \to \mathcal{O} \quad \text{as} \quad t \to \pm\infty.$$

Chaotic motion of this kind occurs in a subset of the state space C which is thin, without interior, and determined by a Cantor set. Numerical simulations since the work of Mackey and Glass [9], however, suggest chaotic solution behaviour all over larger, possibly open sets in state-space.

In order to isolate the impact of state-dependent delay from possible effects due to the shape of f we restrict attention to equations of the form

$$x'(t) = -\alpha\, x(t - d(x_t)) \tag{6}$$

with parameter $\alpha > 0$ and with a delay functional $d : C \supset U_0 \to [0, r]$. For d constant Eq. (6) is reduced to Eq. (5) with $f(\xi) = -\alpha\,\xi$ which is linear, and does not exhibit any complicated solution behaviour.

In [8] we found delay functionals for which Eq. (6) generates chaotic motion of Shilnikov type [15], close to a solution which is homoclinic with respect to the stationary point $0 \in C^1$. Also this kind of chaotic motion is present only in a thin subset of the state space, which here is the solution manifold in C^1.

A concept of chaotic motion on a larger, open set requires a trajectory which is dense in the open set, that is, the trajectory visits every neighbourhood of every point in the open set over and over again, for $t \to \pm\infty$.

In [21] we established a weaker form of this for solutions of Eq. (6) with a suitable delay functional d. The weaker form is specific for delay differential equations and means that for some positive $h < r$ the *short segments*

$$x_{t,h} \in C^1([-h, 0], \mathbb{R}), \quad x_{t,h}(s) = x(t + s) \quad \text{for} \quad -h \leqslant s \leqslant 0,$$

along a certain solution $x : [-r, \infty) \to \mathbb{R}$ are dense in an open subset of the space $C^1([-h, 0], \mathbb{R})$.

This type of result makes precise what one sees in simulations of complicated-looking solutions of various delay differential equations since [9].

The proof of the result on dense short segments begins with a step-by-step construction of both a function $x : [-r, \infty) \to \mathbb{R}$ and a delay function

$$\Delta : [0, \infty) \to [0, r]$$

so that the non-autonomous linear equation

$$x'(t) = -\alpha\, x(t - \Delta(t)) \tag{7}$$

holds for all $t \geqslant 0$ and there exists $h \in (0, r)$ so that the short segments $x_{t,h}$ are dense in some open subset of the space $C^1([-h, 0], \mathbb{R})$. The curve $t \mapsto x_t$ in the space C^1 is injective, and the equation

$$d(x_t) = \Delta(t)$$

turns the delay function Δ into a delay functional along the curve. Upon that d is extended to a continuously differentiable map on an open neighbourhood U of the trace $\{x_t \in C^1 : t \geqslant t_*\}$ in C^1, for some $t_* > 0$. Finally property (e) is verified for the map $f : U \to \mathbb{R}$ given by

$$f(\phi) = -\alpha\, \phi(-d(\phi))$$

so that Eq. (6) is regular in the sense that it generates a nice semiflow on the solution manifold.

Lecture 3: The Impact of State-Dependent Delay on a Periodic Orbit

We begin with a periodic solution of an autonomous differential equation with a constant time lag and ask how stability properties of the periodic solution change when the constant time lag is replaced by a variable, state-dependent delay - in such a way that the periodic solution is preserved. The equation with the constant time lag is

$$x'(t) = g(x(t-1)) \tag{8}$$

with a continuously differentiable odd function $g : \mathbb{R} \to \mathbb{R}$ which is constant on $(-\infty, -b]$ for some $b > 0$ and positive on $(-b, 0)$. It is not difficult to compute explicitly a periodic solution $p : \mathbb{R} \to \mathbb{R}$ of Eq. (8), compare the beginning of [2, Chapter XV]. Eq. (8) shows that actually p is twice continuously differentiable. The period is 4, and p has the symmetry

$$p(t+2) = -p(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

Consider the spaces $C = C([-2,0], \mathbb{R})$ and $C^1 = C^1([-2,0], \mathbb{R})$. Because of its symmetry the function p is a solution of every equation

$$x'(t) = g(x(t - d(p_t)))$$

with $d : C \to [0,2]$ of the form

$$d(\phi) = 1 + \rho(\phi(0) + \phi(-2)),$$

for a function $\rho : \mathbb{R} \to (-1,1)$ satisfying $\rho(0) = 0$. We fix a continuously differentiable function $\delta : \mathbb{R}^2 \to (-1,1)$ with

$$\begin{aligned} \delta(\xi, 0) &= 0 \quad \text{for all} \quad \xi \in \mathbb{R}, \\ \delta(0, \Delta) &= 0 \quad \text{for all} \quad \Delta \in \mathbb{R}, \\ \partial_1 \delta(0, \Delta) &= \Delta \quad \text{for all} \quad \Delta \in \mathbb{R}, \end{aligned}$$

e. g., $\delta(\xi, \Delta) = \sin(\xi\, \Delta)$, or $\delta(\xi, \Delta) = \tanh(\xi\, \Delta)$, and define $d_\Delta : C \to (0,2)$ for $\Delta \in \mathbb{R}$ by

$$d_\Delta(\phi) = 1 + \delta(\phi(0) + \phi(-2), \Delta).$$

Then $d_\Delta(p_t) = 1$ for all $t \in \mathbb{R}$, and p becomes a solution of the equation

$$x'(t) = g(x(t - d_\Delta(x_t))), \tag{9}$$

which for $\Delta = 0$ is Eq. (8) with the constant time lag 1 while for $\Delta \neq 0$ there is a state-dependent contribution to the time lag in the differential equation. For every $\Delta \in \mathbb{R}$ the theory presented in the first lecture applies, and there are continuously differentiable solution operators $S_{\Delta,t}$, $t \geqslant 0$, on the solution manifold X_Δ associated with Eq. (9).

The initial segment p_0 is a fixed point of each period map $S_{\Delta,4}$, $\Delta \in \mathbb{R}$. The stability properties of p as a solution to Eq. (9) which we have in mind are the spectral properties of the linearization

$$DS_{\Delta,4}(p_0) : T_{p_0}X_\Delta \to T_{p_0}X_\Delta,$$

of the period map at its fixed point p_0. The map $M_\Delta = DS_{\Delta,4}(p_0)$ is called monodromy operator, as in analogous scenarios with ordinary differential equations. Recall that

$$M_\Delta \chi = v_4^\chi$$

with the solution $v = v^\chi$ of the variational equation along p, which is

$$v'(t) = g'(p(t-1))\{v(t-1) - p'(t-1)\Delta[v(t) + v(t-2)]\}. \tag{10}$$

From $p(-1) < -b$, hence $g'(p(0-1)) = 0$, we get that the domain and range $T_{p_0}X_\Delta$ of the monodromy operator is

$$Y = \{\chi \in C^1 : \chi'(0) = 0\},$$

which is independent of the parameter Δ. The operators M_Δ are compact, so their spectra, or more precisely, the spectra of their complexifications $\mathcal{M}_\Delta : \mathcal{Y} \to \mathcal{Y}$, are at most countable and consist of eigenvalues of finite algebraic multiplicity. Following conventions for ordinary differential equations we call the eigenvalues of the monodromy operators Floquet multipliers. The number 1 is always a Floquet multiplier, with eigenvector p_0'.

It is not difficult to show that at $\Delta = 0$, where Eq. (9) reduces to Eq. (8), the spectrum σ_Δ of $\mathcal{M}_\Delta$ is simply $\{0, 1\} \subset \mathbb{C}$, 1 is a simple eigenvalue, and 0 is an eigenvalue with geometric eigenspace of codimension 1. On the level of linearization this reflects the fact that the orbit $\mathcal{O} = \{p_t \in C^1 : 0 \leqslant t \leqslant 4\}$ is superstable in the sense that all initial data in a neighbourhood of $\mathcal{O}$ in X_0 define solutions of Eq. (8) whose segments merge into $\mathcal{O}$ in finite time. The result for the case $\Delta = 0$ means that the state-dependent delay for $\Delta > 0$ can only result in some kind of destabilization of the periodic orbit $\mathcal{O}$.

Using the variational equation (10) one finds a characteristic equation for the Floquet multipliers in $\mathbb{C} \setminus \{0, 1\}$ and is able to compute the resolvents $(\mathcal{M}_\Delta - \lambda)^{-1}$, $\lambda \in \rho_\Delta = \mathbb{C} \setminus \sigma_\Delta$. This is inspired by an approach going back to [19] and [16, 17].

Analyzing the characteristic equation one obtains the following results. At $\Delta = 0$ a Floquet multiplier $\Lambda(\Delta) \in \sigma_\Delta \cap (-\infty, 0)$ bifurcates from $0 \in \mathbb{C}$ and decreases to $-\infty$ as $\Delta \to \infty$, with nonzero speed. This means a loss of stability of the periodic orbit $\mathcal{O}$ for $\Delta > 0$; for $\Delta > 0$ with $\Lambda(\Delta) < -1$ the orbit $\mathcal{O}$ is unstable. The Floquet multiplier 1 is simple for all parameters $\Delta \geqslant 0$, and the Floquet multiplier $\Lambda(\Delta)$ is simple for $\Delta = \Delta_*$ with $\Lambda(\Delta_*) = -1$. There are no real Floquet multipliers in the interval $(1, \infty)$. In the interval $(0,1)$ there appear pairs of real Floquet multipliers at a sequence of critical parameters $\Delta_k > 0$ and converge to 1 as $\Delta \to \infty$. Subcritical bifurcations at the critical parameters yield pairs of complex conjugate Floquet multipliers.

In particular all spectral hypotheses for a period-doubling bifurcation from the periodic orbit $\mathcal{O}$ at the critical parameter $\Delta = \Delta_*$ are satisfied. Let us mention that we are not aware of any other example of a period doubling bifurcation in delay differential equations.

As in the case of periodic solutions of ordinary differential equations the Floquet multipliers and their multiplicities should be invariants of the orbit $\mathcal{O} \subset X_\Delta$, which means that they should not change if the solution p of Eq. (9) is replaced with a translate $p(t+\cdot)$, $0 < t < 4$. A proof of this in case of delay differential equations with constant time lags is found in [2, Chapters XIII-XIV].

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